A Constitutive Formulation for High-Elongation Propellants

S. R. Swanson*
University of Utah, Salt Lake City, Utah
and
L. W. Christensen†
Hercules Incorporated, Magna, Utah

A constitutive model is presented that appears to successfully represent the stress-strain behavior of a high-elongation propellant over a wide range of conditions. The model uses a Lagrangian formulation and separates the propellant response into shearing and volumetric components. Particular features of the model are the use of a strain-softening function multiplying a convolution integral that is analogous to that of linear viscoelasticity. Within the integral is an additional term that modifies the history effects. This term appears to correlate behavior at changing strain rate with that at changing temperature. Comparisons with experimental data are given.

Nomenclature

a_T = time-temperature shift factor
B = inverse of time-temperature shift factor
B_n = inverse of time-temperature shift factor of temperature at time t_n
$\bar{B}_n$ = time-averaged value of B_n
C = right Cauchy-Green deformation tensor
C_I = constant used in unloading and reloading strain-softening function
E = Green-St. Venant strain tensor
E_i = exponential series constants in tensile stress relaxation modulus
E'_{ij} = deviatoric components of Green-St. Venant strain tensor
$E_{\rm rel}(t)$ = linear tensile stress relaxation modulus
F = deformation gradient tensor
f = volumetric stress-strain function; also a shear convolution function of time
f_c = shear convolution function of constant deviatoric strain rate
$G_{\rm rel}(t)$ = linear shear stress relaxation modulus
G_i = exponential series constants in shear stress relaxation modulus
$g(\epsilon)$ = strain-softening function
I = unit tensor
$I_{n,i}$ = terms in numerical calculation of convolution integral
$IC_{n,i}$ = terms in numerical calculation of convolution integral for constant deviatoric strain rate
P = pressure
$P_{n,i}$ = term used in numerical calculation of convolution integral
R_ϵ = constant strain rate
S = symmetric Piola-Kirchhoff stress tensor
S' = deviatoric Piola-Kirchhoff stress tensor
T = temperature; transpose
t = time
t_n = time at step n in numerical solution of convolution integral
Δt_n = time increment at step n
v = volume in deformed shape
v_0 = initial volume
X = coordinates in the reference configuration
x = coordinates in the deformed configuration
α = coefficient of thermal expansion
α_i = constants for exponential series representation of linear stress relaxation modulus
β = constant used in modification of convolution integral for changing strain rates
γ = function used in modification of convolution integral
ϵ = infinitesimal strain
$\bar{\epsilon}$ = infinitesimal strain minus thermal expansion
η = dummy variable in temperature-reduced time calculation
λ_I = principal stretch ratio
ξ = temperature-reduced time
ξ_n = value of temperature-reduced time at time t_n
ϵ' = dummy variable for temperature-reduced time
ρ = density in deformed configuration
ρ_0 = density in reference configuration
σ = Cauchy stress
τ = dummy variable for time
ϕ = function-modifying convolution integral for varying strain rate
ψ, ψ_n = effective value of time used in calculation of convolution integral for constant strain rate

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*Associate Professor, Department of Mechanical and Industrial Engineering.
†Technical Specialist.

Introduction

THE stress analysis of solid-propellant grains requires a constitutive, or stress-strain, relationship for the solid-propellant material. Linear elastic or linear viscoelastic models have been widely considered as inadequate models of propellant behavior, except under special loading conditions. Despite the need for more general constitutive relationships, progress in this area has been difficult.

A number of factors appear to increase the need for more general propellant stress-strain laws at this time. For example, the large strain capability of many current propellants (on the order of 100% strain) indicates that at least the kinematics of large deformations must be considered. Further, the use of stresses as failure parameters, as in fracture mechanics, generally places more stringent requirements on the structural analysis than would be the situation if strains were of primary importance. Finally, the general increase in the capabilities of structural computer codes has made it possible and practical to utilize large deformation, nonlinear, material models in a routine analysis.

Several approaches have been explored previously in attempts to develop nonlinear constitutive relations for rate-dependent materials. The multiple integral representation has been studied extensively for a number of materials.[1,2] It has apparently been difficult to reduce this method to practice. It has not been shown that this representation can in fact be used to model the stress-strain behavior of interest here. An alternative approach has been used by Farris,[3] who emphasized the nonlinear behavior of solid propellants even at small strains when unloading and reloading conditions are considered. Farris's approach may be considered as a combination of deformation plasticity with linear viscoelasticity, as elastic-like stress-strain laws are used for conditions where the maximum principal strain is at a lower value than at some previous time in the loading history. This approach has led to an accurate modeling of unloading/reloading conditions, although the fitting of parameters for other conditions has apparently been somewhat difficult.

Schapery[4] has considered the micromechanics of nonlinear behavior in solid propellants, which he postulated to be due to cracking near the filler particles. The crack propagation is then modeled by means of viscoelastic fracture mechanics. This approach appears to give a good model for relatively low strain loading and unloading and is similar to Farris's model in that respect. There is no question that a micromechanics approach provides more insight into material behavior than does a purely phenomenological approach. The difficulty is that the specific mechanisms responsible for material behavior must be identified and modeled. For example, although Schapery considers binder cracking as a source of nonlinear behavior, polymer chain sliding may play an additional role at higher strain levels. Additional mechanisms may raise the complexity of the stress-strain calculation beyond that reasonable for a structural computer code. It would seem that the micromechanics and phenomenological approaches are complementary, with the micromechanics providing more insight and phenomenology being more useful for computations. The use of hidden variables represents a sort of middle ground, in that the hidden variables are usually treated phenomenologically but can have some physical significance. An example of this is the recent work of Quinlan,[5] who considers a "bonding parameter" for uniaxial tension cycling tests.

While each of these approaches has its advantages, it still appears that a constitutive framework for solid-propellant behavior is not readily available. A significant aspect is the difficulty of determining the parameters required for the various models.

Before getting involved in establishing a new model framework, it is worthwhile to consider some typical examples of stress-strain behavior. As an illustration, some stress-strain curves representative of a modern high-elongation propellant are shown in Fig. 1. Principal features are the usual viscoelastic dependence on strain rate, the large strain capability, the dependence of the stress-strain response on superposed pressure, and, as will be shown later, a marked deviation from behavior associated with linear viscoelasticity. This behavior is generally typical of many solid propellants.

In the following, a simple phenomenological stress-strain model is developed that appears to represent to a reasonable degree the mechanical behavior of solid propellants over a wide range of variables. This model contains elements of linear viscoelasticity as well as large-deformation elastic-plastic behavior. The appropriate stress and strain measures are discussed. Finally, the determination of the model parameters is illustrated along with comparisons with experimental data.

Model Formulation

The basic idea to be pursued in this work is that propellant behavior can be represented over a rather wide range of conditions by the stress-strain law given in its one-dimensional stress and infinitesimal deformation form as

$$\sigma(t) = g(\epsilon) \int_0^t E_{\text{rel}}(t-\tau)\phi(\tau)\frac{\partial \epsilon}{\partial \tau}\,\mathrm{d}\tau \tag{1}$$

This equation is recognized as the usual hereditary integral of linear viscoelasticity, but with two additional features: the $\phi(\tau)$ function inside the integral and the multiplying strain function $g(\epsilon)$. The function g can be considered as a strain-softening function, and has been used in this sense by a number of previous investigators.[6] Its use is motivated by the nonlinearity commonly observed in stress relaxation tests as well as the fact that constant strain rate tests at various strain rates often show a similar nonlinearity. The function ϕ has been introduced to account for a marked discrepancy between observed propellant behavior and that predicted by linear viscoelasticity under the conditions of a changing temperature-reduced strain rate. Thus ϕ will be developed so as to have a value of unity for the isothermal constant strain rate or stress relaxation conditions, but will have a significant effect when the temperature-reduced strain rates vary.

The large strain capability of many solid propellants requires that the stress-strain relation be formulated in terms of the appropriate stress and strain tensors. The convolution must be written so that the stress in a material element depends on the deformation history of that element. This is most easily handled by means of a Lagrangian analysis using the symmetric Piola-Kirchhoff stress and the Green-St. Venant (Lagrangian) strain tensors, as both measures consider a material element referred to a specific reference frame that is usually taken as the original unstressed configuration. This substitution of Lagrangian variables has been employed recently by Christensen,[7] for example, and many others. Thus the generalization of Eq. (1) would be

$$S_{xx} = g(E) \int_0^t E_{\text{rel}}(t-\tau)\phi(\tau)\frac{\partial E_{xx}}{\partial \tau}\,\mathrm{d}\tau \tag{2}$$

where E_{xx} is the normal strain component of the Green-St. Venant strain tensor in the x direction and S_{xx} the corresponding component of the symmetric Piola-Kirchhoff stress. The Green-St. Venant strain tensor is defined as[8]

$$E = \tfrac{1}{2}[F^T F - I] \tag{3}$$

where F is the usual deformation gradient given by

$$F_{ij} = \frac{\partial x_i}{\partial \bar{X}_j} \tag{4}$$

and the x_i are coordinates of a material point located at $\bar{X}_i$ in the reference configuration. The Piola-Kirchhoff stress is often called a "pseudostress," but has the advantage that it is measured with respect to the reference configuration. It is related to the more physically meaningful Cauchy or "true" stress σ_{ij} based on the current deformed configuration by the equation

$$\sigma = (\rho/\rho_0)FSF^T \tag{5}$$

A significant difficulty exists in generalizing a uniaxial law of the form of Eq. (2) to three dimensions, when the material is nearly incompressible. "Nearly compressible" means that the bulk modulus associated with volumetric straining is much higher than the shearing modulus associated with distortion. This situation is characteristic of solid propellants, where the moduli can differ by two orders of magnitude. The difficulty is that of separating these two phenomena so that distortional stresses are not overestimated by inadvertently including the bulk stiffness in the calculation. This is particularly important in triaxial loadings.

The change in volume of a material element can be readily calculated from the deformation gradient. Defining the right Cauchy-Green deformation tensor C as

$$C = F^T F \tag{6}$$

the volume change is given as

$$v/v_0 = \sqrt{\det C} = \sqrt{IIIC} \tag{7}$$

where the determinant is an invariant commonly denoted as $\sqrt{IIIC}$. An appropriate volumetric stress-strain relation then involves the mean Cauchy stress and the volumetric strain as

$$\sigma_{kk}/3 = f(\sqrt{IIIC} - 1) \tag{8}$$

where f is a possibly nonlinear relation reducing to the bulk modulus times $(\sqrt{IIIC} - 1)$ for small strains. When material failure occurs, such as by dewetting, f may also have to include deviatoric stresses or strains. In any case, the volumetric stress-strain relationship inherently involves the Cauchy stress if the volumetric relationship is to be independent of distortional strain, as it is usually assumed to be when dewetting is not involved. This same situation is seen in, for example, rubber elasticity where, although the Piola-Kirchhoff stress is related to the Green-St. Venant strain through the derivatives of the strain energy function, the mean stress is determined in terms of Cauchy stress.[9]

If Eq. (8) is taken as defining the mean stress, the remaining task is to establish a relationship between distortional stress and strain. It may be recalled that this is accomplished by deviatoric stresses and strains in small-strain theory. To this end we define deviatoric components of the symmetric Piola-Kirchhoff stress by the relation

$$S' = S - (S_{kk}/3)I \tag{9}$$

The deviatoric components S' of the Piola-Kirchhoff stress will now be assumed to be related to the deviatoric components of the Green-St. Venant strain. For lack of better data at this point, we assume that

$$S'_{ij} = g(\epsilon)\int_0^t 2G_{\text{rel}}(t-\tau)\phi(\tau)\frac{\partial E'_{ij}}{\partial \tau}\,\mathrm{d}\tau \tag{10}$$

which (for g and ϕ equal to unity) reduces to isotropic, linear viscoelasticity for small strain. In the above, E' has been defined as usual by

$$E'_{ij} = E_{ij} - (E_{kk}/3)I \tag{11}$$

Finally, we require that the Cauchy stress given by

$$\sigma = \frac{\rho}{\rho_0}FSF^T = \frac{\rho}{\rho_0}FS'F^T + \frac{\rho}{\rho_0}\frac{S_{kk}}{3}FIF^T \tag{12}$$

satisfy the mean stress/volume strain relationship given previously in Eq. (8), so that S_{kk} can be determined from

$$\frac{Tr\sigma}{3} = \frac{\rho}{\rho_0}Tr(FS'F^T) + \frac{\rho}{\rho_0}\frac{S_{kk}}{3}TrFF^T = f(\sqrt{IIIC} - 1) \tag{13}$$

or

$$\frac{S_{kk}}{3} = \frac{[(\rho_0/\rho)f(\sqrt{IIIC}-1) - Tr(FS'F^T)]}{TrFF^T} \tag{14}$$

It may be noted that S' and E' do not have the same physical significance as their small-strain counterparts. For small strain the use of deviatoric components separates bulk from distortional effects. For large strain this is not true; for example, a volumetric strain superposed over a distortional strain state will change the values of E' slightly. However, as long as the bulk modulus is large relative to the shear modulus, this effect is small. In the cases investigated here, the discrepancy was not numerically significant. A similar conclusion was reached by Ogden[10] for nearly incompressible rubber, but with a different stress-strain law.

Application to High-Elongation Propellants

The stress-strain law expressed in Eq. (10) will now be applied to the high-elongation propellant data given previously. Additionally, the g and ϕ functions will be developed.

First, consider fitting the model to the usual constant crosshead speed uniaxial tensile tests. For these purposes the deformation can be based on the assumption of incompressible behavior (for fitting the shearing functions only) so that the Cauchy stress is given by

$$\sigma_{11} = \frac{g(\epsilon)(2\lambda_1^2 + 1/\lambda_1)}{3}\int_0^t G_{\text{rel}}(t-\tau)\phi(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\lambda_1^2 - \frac{1}{\lambda_1}\right)\mathrm{d}\tau \tag{15}$$

Taking the rate of change of $(\lambda_1^2 - 1/\lambda_1)$ as being approximately constant and taking ϕ as unity (for constant strain rate) then gives

$$\frac{\sigma_{11}}{[g(\epsilon)(2\lambda_1^2 + 1/\lambda_1)/3]} = 3\dot{\lambda}_1\int_0^t G_{\text{rel}}(t-\tau)\,\mathrm{d}\tau \tag{16}$$

The term in the right-hand side of Eq. (16) is shown compared with the Cauchy stress measured for a propellant tensile test in Fig. 2. The relaxation modulus actually used was the tensile relaxation modulus measured at approximately 1% strain with the approximation $3G_{\text{rel}}(t) \approx E_{\text{rel}}(t)$. The g function is then determined for loading conditions from the ratio of these curves and Eq. (16). A typical curve is shown in Fig. 3. This function can then be numerically approximated by suitable polynomial and/or exponential functions. The effect of pressure has also been incorporated into this function by making the parameters of the curve fit a function of the mean Cauchy stress if the mean stress is compressive. A typical curve is shown in Fig. 3. Unloading hysteresis and deformation plasticity can also be incorporated into this function by appropriate changes. It is assumed that the strain function g may have to be expressed in terms of the strain invariants such as $\sqrt{IIE'}$, the second invariant of the deviatoric Green-St. Venant strain that is analogous to the Von Mises strain of

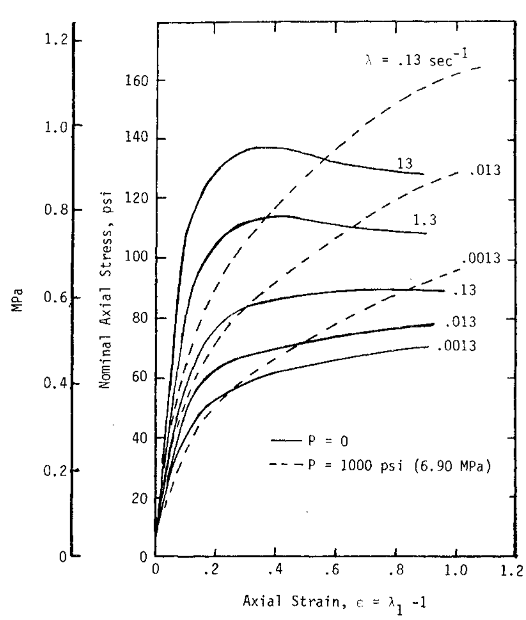

Fig. 1 Stress-strain curves for high-elongation propellant in uniaxial tension.

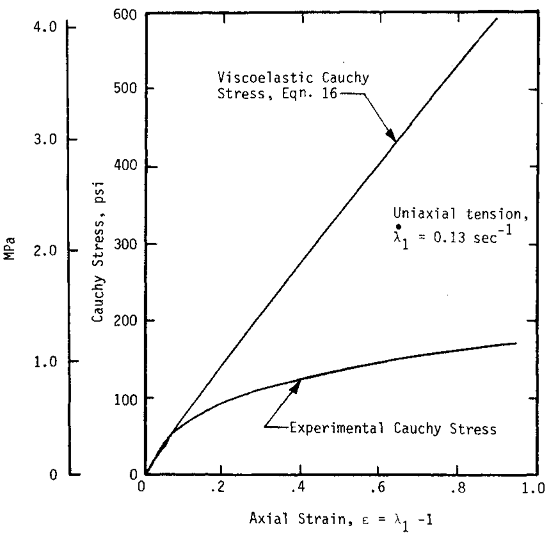

Fig. 2 Comparison of measured and viscoelastic [Eq. (16)] Cauchy stress.

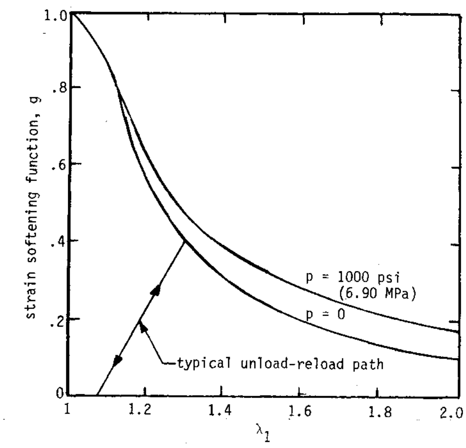

Fig. 3 Effect of deformation and pressure on the strain-softening function g.

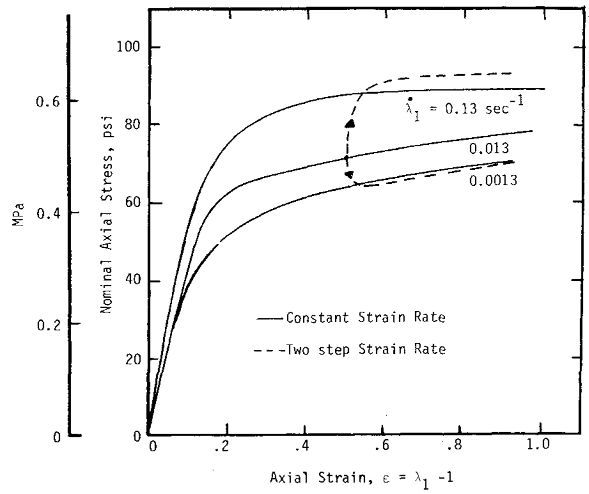

Fig. 4 Measured stress-strain response in changing strain rate uniaxial tension tests.

plasticity strain-hardening functions. However, little data are available at this point to suggest a particular choice.

The algorithm developed by Herrmann and Peterson[11] has been used to implement the calculation of the convolution integral. In brief, let the shear relaxation modulus be represented by a Prony series as

$$G_{\mathrm{rel}}(t) = \sum_{i=1}^{m} G_i \exp(-\alpha_i t) \tag{17}$$

If we define the convolution at a particular time t_n as

$$f(t_n) = \int_0^{t_n} G_{\mathrm{rel}}(t_n - \tau) \frac{\partial E'}{\partial \tau} \mathrm{d}\tau \tag{18}$$

then, as shown in Ref. 11, a recursion relation can be easily developed to compute $f(t_n)$. Let

$$f(t_n) = \sum_{i=1}^{m} I_{n,i} \tag{19}$$

and

$$I_{n,i} \equiv \int_0^{t_n} G_i \exp[-\alpha_i (t_n - \tau)] \frac{\partial E'}{\partial \tau} \mathrm{d}\tau \tag{20}$$

then

$$I_{n,i} = \exp(-\alpha_i \Delta t_n) I_{n-1,i} + \dot{E}'_n G_i [1 - \exp(-\alpha_i \Delta t_n)]/\alpha_i \tag{21}$$

which gives for the change in these terms

$$\Delta I_{n,i} = \dot{E}'_n G_i [1 - \exp(-\alpha_i \Delta t_n)]/\alpha_i - I_{n-1,i}[1 - \exp(-\alpha_i \Delta t_n)] \tag{22}$$

It has been mentioned previously that the ϕ function of Eqs. (1) and (10) is to be taken as unity for constant or zero strain rates. For the conditions of a changing strain rate, the experimental results shown in Fig. 4 appear to necessitate a different treatment. Shown in this figure are tests in which the nominal strain rate (i.e., crosshead speed) is changed by a factor of 10 during the test. As can be seen, the experimentaly measured stress changes rather quickly to the level corresponding to a test at the second strain rate. In contrast, the linear viscoelastic prediction would have a much longer transition. Thus, there is a significant difference between the experiment and the rate dependence associated with linear viscoelasticity for conditions of changing strain rate.

It is believed that the above phenomena are significant and should be emphasized. One important manifestation of this behavior may be in combined straining and cooling conditions. Although the conditions discussed thus far have been isothermal, it is an obvious generalization to consider the time scale in the sense of temperature-reduced time, with the usual assumption of thermorheologically simple behavior. Thus, cooling and straining would correspond to steadily increasing equivalent strain rate. This may explain why stresses are usually significantly underpredicted by linear viscoelasticity in this situation. An example will be given subsequently.

The ϕ function is the heredity integral is developed as follows. Define the response of the heredity integral to a constant strain rate as f_c, given by

$$f_c(t) = R_\epsilon \int_0^{\psi} G_{\mathrm{rel}}(\psi - \tau) \mathrm{d}\tau \tag{23}$$

where $\psi = E'/R_\epsilon$. The general form of $\phi(\tau)$ is

$$\phi(\tau) = 1 + \gamma[f_c(\tau) - f(\tau)] \tag{24}$$

with $\gamma(0) = 0$. For the case where a Prony series representation of the relaxation modulus is used in conjunction with the algorithm of Ref. 11 as described above, an especially convenient form for ϕ is as follows. As above, but including ϕ, let

$$f(t_n) = \int_0^{t_n} G_{\mathrm{rel}}(t_n - \tau) \frac{\partial E'(\tau)}{\partial \tau} \phi \mathrm{d}\tau = \sum_{i=1}^{M} I_{n,i} \tag{25}$$

where

$$I_{n,i}=\int_0^{t_n} G_i\exp[(-\alpha_i(t_n-\tau)]\frac{\partial E'(\tau)}{\partial\tau}\phi_i(\tau)\mathrm{d}\tau \qquad (26)$$

The recursion relation is then

$$I_{n,i}=I_{n-1,i}\exp(-\alpha_i\Delta t_n)+P_{n,i}\phi_{n,i} \qquad (27)$$

where for convenience the definition

$$P_{n,i}=\dot{E}'_n G_i\frac{[1-\exp(-\alpha_i\Delta t_n)]}{\alpha_i} \qquad (28)$$

has been used. Then let $\phi_{n,i}$ be given as

$$\phi_{n,i}=1+\frac{(IC_{n-1,i}-I_{n-1,i})\exp(-\alpha_i\Delta t_n)\beta}{P_{n,i}} \qquad (29)$$

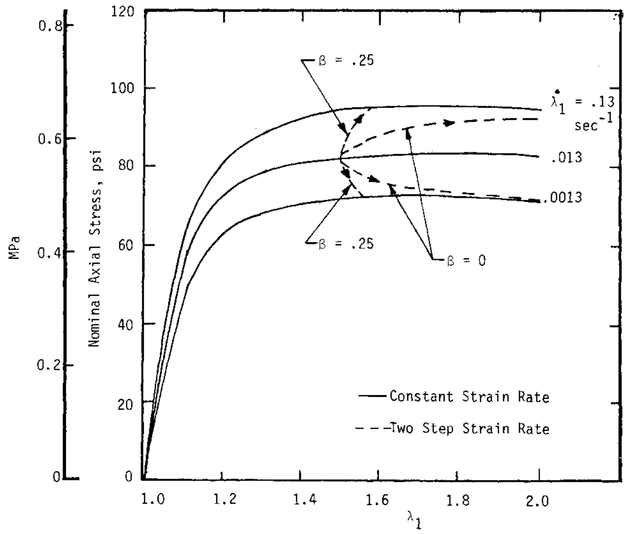

Fig. 5 Effect of model parameter β in response to changing strain rate.

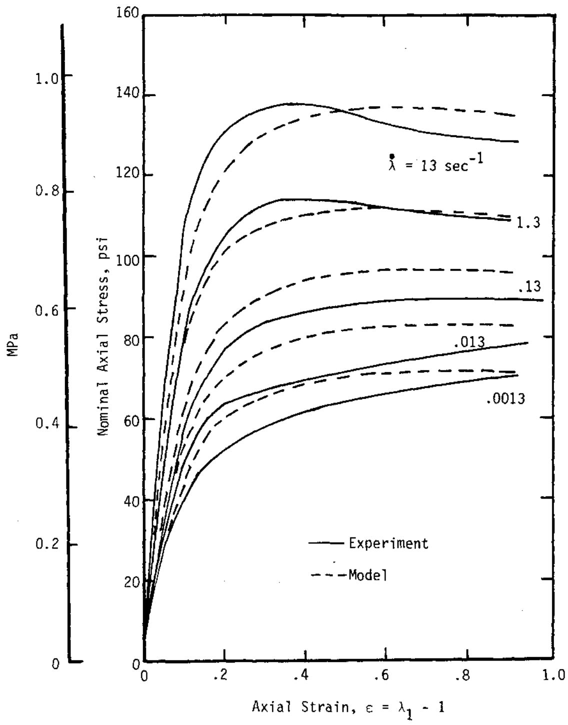

Fig. 6 Comparison of model and experiment, stress-strain response in uniaxial tension.

where β is a constant between 0 and 1 and $IC_{n,i}$ the response to a constant strain rate given by

$$IC_{n,i}=R_\epsilon\int_0^{\psi_n} G_i\exp[-\alpha_i(\psi_n-\tau)]\mathrm{d}\tau$$

$$=R_\epsilon G_i\frac{[1-\exp(-\alpha_i\psi_n)]}{\alpha_i} \qquad (30)$$

and

$$\psi_n=E'_n/R_\epsilon$$

The effect of this definition of ϕ on the response to changes in the strain rate is to give behavior analogous to linear viscoelasticity for $\beta=0$ (thus $\phi=1$). On the other hand, when $\beta=1$ the stress jumps instantaneously to the value that would have occurred had the strain rate been at its current level throughout the deformation history. Finally, intermediate values of β give a transition behavior that is close to that exhibited by propellants. An example is shown in Fig. 5. It can be seen that β near 0.25 gives good agreement with the experimental data.

The stress-strain behavior calculated from the above is shown in Fig. 6 for constant crosshead speed tension tests and Fig. 7 for changing test speeds. It can be seen that the experimental data are well represented.

Unloading tests require a further refinement of the model. For lack of more detailed information, the parameter β was taken as zero for unloading states, i.e., states in which $\sqrt{IIE'}$ is decreasing. The large amount of hysteresis seen in load/unload cycles was then modeled in part by the hysteresis inherent in linear viscoelasticity, but primarily through the g function. This can be accomplished by giving g a different value when the deformation invariant $\sqrt{IIE'}$ has a value less than its maximum previously achieved during the loading history. If we call this current maximum value $\sqrt{IIE'_{max}}$, the function

$$g=g(\sqrt{IIE'_{\max}})\left\{1-C_1\left[1-\frac{\sqrt{IIE'}}{\sqrt{IIE'_{\max}}}\right]\right\} \qquad (31)$$

provides plasticity-like behavior, as illustrated in Fig. 8.

The behavior of the g function for unloading and reloading conditions is illustrated in Fig. 3. An interesting comparison with experimental data is shown in Fig. 9 for a uniaxial tension test in which the extension ratio was a sine function of time. Thus, both changing strain rate and loading/unloading conditions were imposed.

Application to Combined Straining and Cooling

It is straightforward to incorporate changing temperature into the above constitutive formulation. This will be

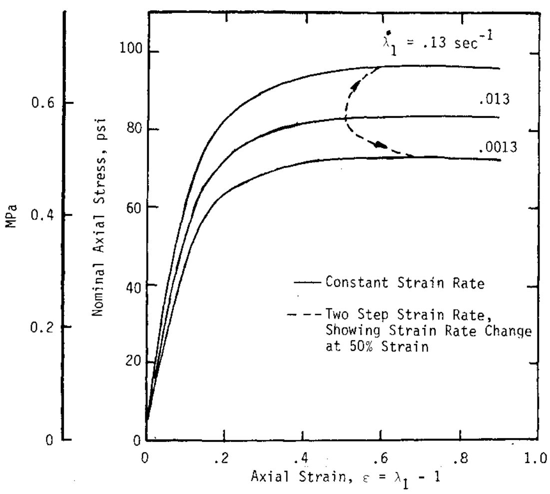

Fig. 7 Model prediction for changing strain rate.

illustrated for the one-dimensional stress, infinitesimal deformation case to facilitate a comparison with experimental results given in Ref. 12, although the three-dimensional, Lagrangian analysis from above could be used with no difficulty.

The stress-strain law to be employed is then

$$\sigma(t)=g(\epsilon)\int_0^t E_{\rm rel}(\xi-\xi')\phi\frac{\partial\bar{\epsilon}}{\partial\tau}\mathrm{d}\tau \tag{32}$$

where thermorheologically simple behavior is assumed, and the reduced time ξ is given by

$$\xi=\int_0^t\frac{\mathrm{d}\eta}{a_T(\eta)}\qquad \xi'=\int_0^\tau\frac{\mathrm{d}\eta}{a_T(\eta)} \tag{33}$$

and $\bar{\epsilon}=\epsilon-\alpha\Delta T$. Using the exponential series representation for the relaxation modulus and employing the notation used previously

$$\sigma(t_n)=g(\epsilon)\sum_{i=1}^{M}I_{n,i} \tag{34}$$

where

$$I_{n,i}=\int_0^{t_n}E_i\exp[-\alpha_i(\xi_n-\xi')]\phi(\tau)\dot{\epsilon}(\tau)\mathrm{d}\tau \tag{35}$$

and, as before

$$I_{n,i}=\exp(-\alpha_i B_n\Delta t_n)I_{n-1,i}$$
$$+\dot{E}_n\phi_n E_i\exp(-\alpha_i\xi_n)\int_{t_{n-1}}^{t_n}\exp(\alpha_i\xi')\mathrm{d}\tau \tag{36}$$

where for convenience $B\equiv 1/a_T$ and B_n is the reciprocal of the shift factor for the temperature existing in time t_n. The integral on the right of Eq. (36) is approximated by using $\xi'=\bar{B}_n\tau$ in the time interval Δt_n. Here $\bar{B}_n$ is an effective value for this time integral found by

$$\bar{B}_n=\left(\frac{1}{t_n}\right)\int_0^{t_n}B(\tau)\mathrm{d}\tau \tag{37}$$

The recursion relation for the components of the hereditary integral then becomes

$$I_{n,i}=\exp(-\alpha_i B_n\Delta t_n)I_{n-1,i}+P_{n,i}\phi_{n,i} \tag{38}$$

where

$$P_{n,i}\equiv\dot{\epsilon}_n E_i\frac{[1-\exp(-\alpha_i\bar{B}_n\Delta t_n)]}{(\alpha_i\bar{B}_n)} \tag{39}$$

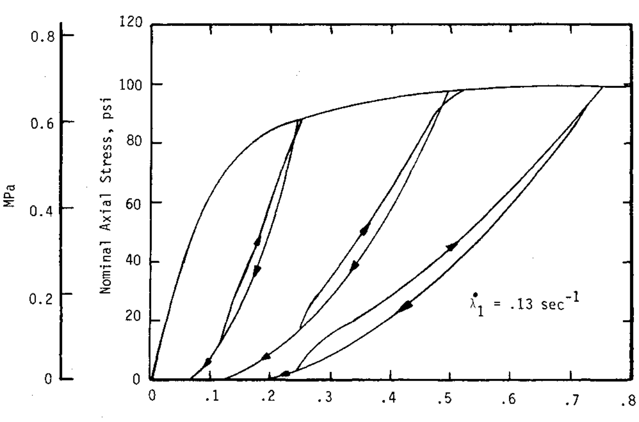

Fig. 8 Model prediction for unloading and reloading in uniaxial tension.

A recursion relation can be easily derived to compute $\bar{B}_n$ as follows. Using the definition of $\bar{B}_n$ above and splitting the time integral into two parts as

$$\bar{B}_n=\left[\left(\frac{1}{t_{n-1}}\right)\int_0^{t_{n-1}}B(\tau)\mathrm{d}\tau\right]\left(\frac{t_{n-1}}{t_n}\right)+\left(\frac{1}{t_n}\right)\int_{t_{n-1}}^{t_n}B(\tau)\mathrm{d}\tau \tag{40}$$

or

$$\bar{B}_n=\left(\frac{t_{n-1}}{t_n}\right)\bar{B}_{n-1}+\left(\frac{\Delta t_n}{t_n}\right)B_n \tag{41}$$

where B_n is assumed constant over the time period Δt_n. Thus $\bar{B}_n$ can be computed numerically using this recursion relation and the current value of the shift factor. The above has assumed the temperature constant over the time interval Δt so that the integrals could be carried out exactly. A linearly varying temperature was used in Ref. 11, along with Gauss numerical integration.

As before, the function ϕ will depend on the difference between the current value of the components of the heredity integral and the value that would obtain if the current strain level had been reached by a strain rate and temperature constant at their current values. Thus defining

$$IC_{n,i}=\bar{R}_\epsilon\int_0^{\psi_n}E_i\exp[-\alpha_i B_n(\psi_n-\tau)]\mathrm{d}\tau$$
$$=\bar{R}_\epsilon E_i\frac{[1-\exp(-\alpha_i B_n\psi_n)]}{(\alpha_i B_n)} \tag{42}$$

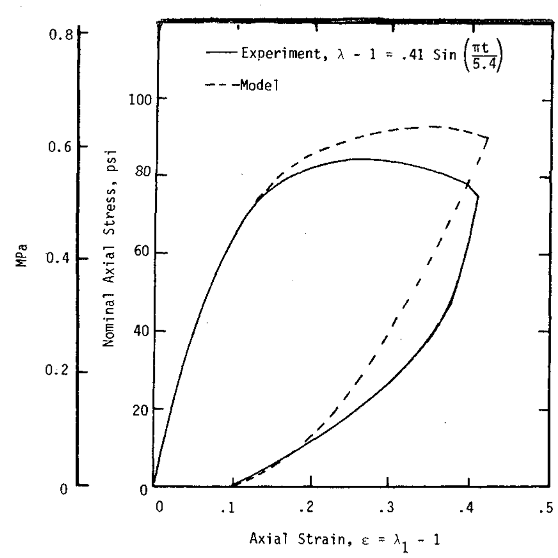

Fig. 9 Comparison of model and experiment for sinusoidal strain input in uniaxial tension.

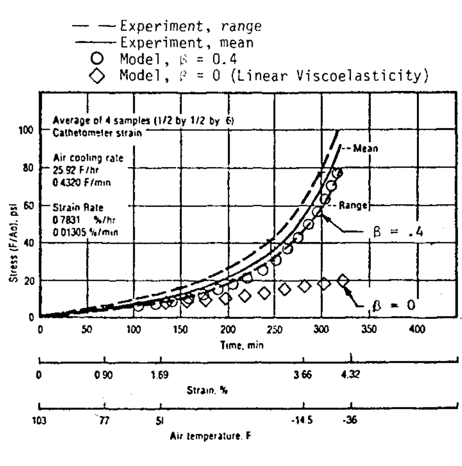

Fig. 10 Comparison of model and experiment (from Ref. 12) for propellant straining and cooling test.

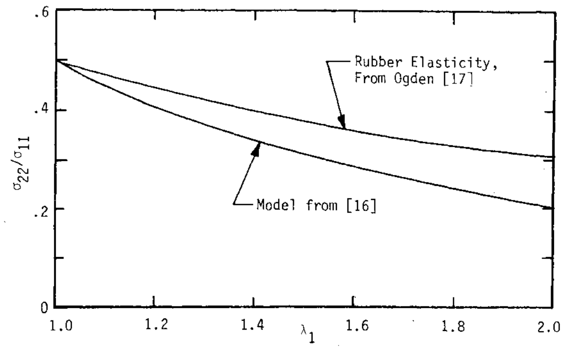

Fig. 11 Comparison of present model with rubber elasticity (from Ref. 17) for stress ratio in biaxial strip test.

Table 1 Relaxation modulus coefficients for straining and cooling example

$$E_{\rm rel}(t) = \sum_{i=1}^{15} E_i \exp(-\alpha_i t)$$

α_i, min^{-1}	E_i, psi	α_i, min^{-1}	E_i, psi
1.E−7	266.15	1.E+1	200.22
1.E−6	−116.14	1.E+2	420.82
1.E−5	76.56	1.E+3	506.24
1.E−4	5.23	1.E+4	1603.25
1.E−3	45.03	1.E+5	1893.33
1.E−2	54.05	1.E+6	5301.66
1.E−1	74.30	1.E+7	4409.99
1.E+0	186.68		

where

$$\psi_n = \bar{\epsilon}_n / \bar{R}_\epsilon$$

with these definitions,

$$\phi_{n,i} = 1 + \frac{(IC_{n-1,i} - I_{n-1,i})\exp(-\alpha_i B_n \Delta t_n)\beta}{P_{n,i}} \tag{43}$$

where as before β is a constant varying between 0 and 1. With $\beta = 0$, the calculation of the integral reduces to that of thermorheologically simple linear viscoelasticity. With $\beta = 1$, the components of the integral assume the values given by linear viscoelasticity if the current value of strain rate and temperature had been constant throughout the history.

Combined cooling and straining test results for uniaxial specimens have been reported in Ref. 12, along with other material properties. A calculation was made to compare with the reported results. The relaxation modulus of Ref. 12 (Fig. 126) was first fitted with a Prony series using a least squares technique. The coefficients obtained are given in Table 1. The shift factor used was that given in Ref. 13 (p. 395) as

$$\frac{1}{a_T} = \left(\frac{\text{Temp} + 100}{76 + 100}\right)^{13.7} \tag{44}$$

where the temperature is in °F.

For lack of better information, the strain-softening factor $g(\epsilon)$ was taken as unity.

The results calculated using Eqs. (36-43) are compared with the experimental results for one of the tests in Fig. 10. It can be seen that the stress predicted by linear viscoelasticity ($\beta = 0$) is low by up to a factor of four. However, good agreement is seen for values of $\beta = 0.4$ and higher. In fact, the best agreement is seen for $\beta = 1$; however, the increase in stress over that obtained for $\beta = 0.4$ is only about 6%. Intermediate values of β, such as used previously to account for changing strain rates (not shown), give reasonable agreement.

Discussion

One of the significant attributes of the present constitutive formulation is the relatively easy parameter determination. Inputs to the model are the linear viscoelastic properties for shear and volume change, a strain-softening and pressure-hardening function g, and a history effect related constant β. A nonlinear volume change relation may be employed if required.

It is surprising that apparently complex behavior can be represented by a relatively simple relationship for g, which has been taken in the present work to be a function only of strain, and the mean stress. However, the work of Schapery[14] on internal damage suggests that in some cases stresses may also have to be considered in this function. Also the unloading behavior reported for some propellants by Farris[3] may indicate that modifications of the g function used at present might be required. However, the basic idea of a multiplying function appears to offer much in the way of simplicity.

The coupling of varying strain rate effects at constant temperature with varying temperature behavior appears to be unique to the present investigation, and appears to work quite successfully. As pointed out in Ref. 13 and also shown above, the cooling and straining situation is one in which linear viscoelasticity gives grossly inaccurate answers. Clearly more experimental work is required to establish the adequacy of the present formulation with respect to various straining and temperature histories. It is also interesting to note that in the context of the present model, setting the parameter $\beta = 1$ gave the best fit. This is equivalent to making the stress a function of only the current strain, strain rate, and temperature. Thus, at least for some conditions, an ad-hoc model of this kind could be successfully employed. This procedure has been used previously by Anderson.[15]

The present model formulation treats multiaxial effects by separating shearing and bulk stiffness. While a plausible framework has been suggested, other possibilities exist. Clearly more experimental data on multiaxial effects are required.

In implementation of the algorithm used for calculating the convolution integral, one value must be updated and saved for each coefficient of the Prony series multiplied by the number of stresses. Thus, in an axisymmetric analysis with 4 stress components, and say for example that 8 terms in the Prony series are used, then 32 values must be updated and saved. If used in conjunction with a finite element computation, then of course this applies for each element. For some large-scale calculations this may be considered excessive. An approximation was presented in Ref. 16 to reduce this to just one set of history terms per element, independent of the number of stress components. This was accomplished using essentially the development presented here, but postulating that a shearing invariant of the stresses depended on the history of an invariant of the deformation. The individual stress components were then obtained from the stress invariant. While this procedure minimizes the computer storage required for the stress-strain model, it does introduce particular assumptions about material behavior. For example, as an indication of multiaxial effects, the ratio of principal Cauchy stresses σ_{22}/σ_{11} in a strip biaxial test has been calculated for the invariant model of Ref. 16 and is shown in Fig. 11. Also shown is the stress ratio predicted by the rubber elasticity model of Ogden.[17] Both models show a decrease in this stress ratio from the linear elasticity value of 0.5 (for $\nu \simeq 0.5$) with increasing extension ratio. While this comparison is interesting, there is no reason why the models should agree. Rather each should match experimental data for the particular material. In the present case, the necessary experimental data to justify a more involved approach were not available.

Summary and Conclusions

A stress-strain model to represent the mechanical behavior of solid propellant has been presented. The model is strictly phenomenologically based, but appears to represent experimental behavior over a wide range of conditions. The model uses a Lagrangian formulation and separates the propellant response into shearing and volumetric components. Nonlinearity is modeled by a multiplicative strain-softening term. Additional history effects are introduced by a term within the convolution integral that appears to correlate propellant behavior at changing strain rates and changing temperature histories.

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